Instantons and 2d Superconformal field theory

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I. Seiberg-Witten theory

In 1994 Seiberg-Witten proposed an exact expression for the low energy effective action (prepotential $\mathcal{F}$)of $\mathcal{N}=2$ SUSY $d=4$ Yang-Mills gauge theory with spontaneous breaking of non-abelian symmetry $SU(2)->U(1)$

$$L_{N=2}=tr\left\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+2\mathcal{D}_{\mu}\Phi^{*}\mathcal{D}^{\mu}\Phi+\right.$$

$$\left.+\sum_{a}(i\bar{\lambda}_{a}\bar{\sigma}_{\mu}\mathcal{D}^{\mu}\lambda^{a}+g\Phi^{*}[\lambda_{a},\lambda^{a}]+g\Phi[\bar{\lambda}_{a},\bar{\lambda}^{a}])+2g^{2}[\Phi^{*},\Phi]^{2})\right\}$$

Using non-renormalization theorem,holomophicity,electric-magnetic duality and some additional physical assumptions like the conjecture about the connection the analytic properties of the prepotential and vanishing masses of dyons S-W write down the explicit expression for $\mathcal{F}$

$$\mathcal{F}(\Psi)=\frac{i}{2\pi}\Psi^{2}\log\frac{\Psi^{2}}{\Lambda^{2}}-\frac{i}{\pi}\sum_{n=1}^{\infty}\mathcal{F}_{N}\left(\frac{\Lambda}{\Psi}\right)^{4N}\Psi^{2}$$

II. Multi-Instanton calculus of DKM+H

To verify this proposal Dorey-Khoze-Mattis started the direct computation coefficients $\mathcal{F}_N$ quasiclassicaly.They get that N-instanton contribution is

$$\mathcal{F}_N = \int d\mu^{(n)} e^{-S_{ind}^{(N)}}$$

$$S_{ind}^{(N)} = \int d^4x tr \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\mathcal{D}_\mu \Phi^* \mathcal{D}^\mu \phi + g\Phi^* [\lambda_a, \lambda^a] \right\}$$

The fields satisfy a reduced set of eq-s of motion

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu};$$

$$\bar{\sigma}_\mu \mathcal{D}^\mu \lambda^a = 0;$$

$$\mathcal{D}_\mu \mathcal{D}^\mu \Phi = -g[\lambda_a, \lambda^a]$$

III. Instantons

Instantons in Yang-Mills theory were discovered by BPST in 1975. They are solutions of the self-duality equation

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu}$$

Using the results of BZ and AW (1977) the general N-instanton solution was constructed by ADHM(1978).
It is expressed through $N \times N$ matrices B_1, B_2, a $N \times 2$ matrix I and a $2 \times N$ matrix J, which obey

$$[B_1, B_2] + IJ = 0$$

$$[B_1 B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0$$

and can be conveniently organizied to $2N \times (2N+2)$ matrix Δ

$$\Delta = a + bz = \begin{pmatrix} I & B_1 & B_2 \\ J^\dagger & -B_2^\dagger & B_1^\dagger . \end{pmatrix} + \begin{pmatrix} 0 & z_1 & z_2 \\ 0 & -\bar{z}_2 & \bar{z}_1 \end{pmatrix}$$

Then the solution is expressed via $2N \times 2$ matrix $U(x)$:
$A_\mu = U^\dagger(x)\partial_\mu U(x)$, if $U^\dagger U = 1$ and $\Delta^\dagger U = 0$,

Due to appearence Weyl zero modes of positive chirality in selfdual Y-M background $S_{ind}^{(N)}$ depends also on grassmann matrices $N \times N$ matrices M_1, M_2, a $N \times 2$ matrix μ and a $2 \times N$ matrix ν which obey

$$[M_1, B_2] + [B_1, M_2] + \mu J + I\nu = 0$$
$$[M_1, B_1^\dagger] + [M_2, B_2^\dagger] + \mu I^\dagger - J^\dagger \nu = 0$$

Taking into account that elements of cotangent bundle $(DB_1, DB_2, D\mu, D\nu)$ satisfy to the same eq-s after the change

$$(M_1, M_2, \mu, \nu) -> (DB_1, DB_2, D\mu, D\nu)$$

DKM transform the integral over super moduli space to integral of the exponential of an mixed differential form.

$$\mathcal{F}_N = \int_{\mathcal{M}_N} e^{-S_{ind}^{(N)}(A, \mathcal{D}A)}$$

IV. Nekrasov patition function

Flume-Poghossian proved that $S_{ind}^{(N)}(A, \mathcal{D}A)$ is an exact equivariant form

$$S_{ind}^{(N)}(A, \mathcal{D}A) = d_v \omega$$

and obtain

$$\mathcal{F}_N = \int_{\mathcal{M}_N} e^{-d_v \omega}$$

Using the localization technique the computation of the integral is reduced to finding fixed points of the vector field and its determinants in these fixed points .

The result is Nekrasov explicit formula for coefficients of S-W prepotential in the pure $\mathcal{N} = 2$ SUSY Yang-Mills theory

$$\mathcal{F}_N = \sum_P \frac{1}{\det v(P)}$$

V. Conformal Field Theory

Two-dimensional conformal Liouville field theory arises in non-critical String theory. The Lagrangian of the theory reads

$$\mathcal{L}_{\mathsf{LFT}} = \frac{1}{8\pi}(\partial_a \phi)^2 + \mu e^{2b\phi}$$

Here μ is the cosmological constant and parameter b is related to the central charge c of the Virasoro algebra

$$c = 1 + 6Q^2 \, , \;\; Q = b + \frac{1}{b} \, .$$

Virasoro algebra

$$[L_m, L_n] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m} \, ,$$

The key problem in CFT is the computation of the correlation functions of the primary fields Φ_Δ . Here Δ is the conformal dimension

$$\Delta(\lambda) = \frac{Q^2}{4} - \lambda^2.$$

4-point correlation function of bosonic primaries Φ_i is expressed in terms so-called Conformal blocks

$$\langle \Phi_1(q) \Phi_2(0) \Phi_3(1) \Phi_4(\infty) \rangle =$$

$$= (q\bar{q})^{\Delta - \Delta_1 - \Delta_2} \sum_{\Delta} C_{12}^{\Delta} C_{34}^{\Delta} F(\Delta_i|\Delta|q) F(\Delta_i|\Delta|\bar{q})$$

This function $F(\Delta_i|\Delta|q)$ is defined in Virasoro representation theory terms uniquely.

Verma module generated by $|\Delta\rangle$ such that

$$L_0|\Delta\rangle = \Delta L_0|\Delta\rangle, \quad L_k|\Delta\rangle = 0, \quad \text{for}\, k > 0$$

The sequence of vectors $|N\rangle_{\Delta_1,\Delta_2,\Delta}$ defined as

$$L_0|N\rangle_{\Delta_1,\Delta_2,\Delta} = (\Delta + N)|N\rangle_{\Delta_1,\Delta_2,\Delta},$$
$$L_k|N\rangle_{\Delta_1,\Delta_2,\Delta} = (\Delta + k\Delta_1 - \Delta_2 + k)|N-k\rangle_{\Delta_1,\Delta_2,\Delta}, \quad k > 0$$

conformal block:

$$F_\Delta(\Delta_1, \Delta_2, \Delta_3, \Delta_4|q) = \sum_N {}_{\Delta_1,\Delta_2,\Delta}\langle N|N\rangle_{\Delta_3,\Delta_4,\Delta} q^N$$

VI. AGT correspondence between $\mathcal{N}=2$ SUSY $4d$ gauge theory and $2d$ Conformal field theory.

Using this explicit expression for the Nekrasov instanton partition function Alday,Gaiotto,Tachikawa in 2009 proposed the remarkable correspondence between this gauge theory and CFT.

In particular they argued that Instanton partition function of 4-dimensional $\mathcal{N}=2$ SUSY gauge theory with four hypemultiplets and four-point Conformal block of 2-dimensional Liouville field theory coincide.

Introduction

During the last years AGT correspondence between $2d$ Conformal field theories and $\mathcal{N}=2$ SUSY $4d$ Gauge theories was extended for cases of CFT with such symmetries, as affine Lie algebras, $\mathcal{W}$ algebras et cet.

It was shown recently that the instanton calculus in the gauge theories on $\mathbb{R}^4/\mathbb{Z}_2$ give rise to the super-Virasoro conformal blocks. The idea to use the $\mathbb{Z}_2$ symmetric instanton moduli $\mathcal{M}_{\text{sym}}$ is based on its conjectured relation to the coset $\widehat{gl}(n)_2/\widehat{gl}(n-2)_2$ which is isomorphic to $\mathcal{A}=\widehat{gl}(2)_2\times\mathcal{NSR}$.

This relation asserts that the algebra $\mathcal{A}$ has a representation with basis labeled by fixed points for the torus action on the moduli spaces $\mathcal{M}_{\text{sym}}$.

We confirm this conjecture by showing that the number of fixed points actually coincides with the number of states in the certain representation of $\mathcal{A}=\widehat{gl}(2)_2\times\mathcal{NSR}$.

Generalizing AGT construction we restrict the equivariant integration by two connected components of $\mathbb{Z}_2$ invariant subspace $\mathcal{M}_{\mathsf{sym}}$ of ADHM moduli space.
The obtained in such a way some new Instanton partition function is supposed to coincide with super-Liouville conformal block up to additional factor related to the algebra $\widehat{gl}(2)_2$..

The so-called Whittaker limit of the Super conformal block is equal to the $\mathbb{Z}_2$ restricted instanton partition function of $SU(2)$ $\mathcal{N}=2$ SUSY pure gauge theory.

To construct the explicit representation of the four-point super-Liouville conformal block function for Neveu-Schwatz sector we take into account that the integral over the moduli space involves also integration over zero modes of the matter fermions.

The instanton partition function is checked to coincide with the four-point conformal block

Moduli space

ADHM date consist of $N \times N$ matrices B_1, B_2, a $N \times 2$ matrix I and a $2 \times N$ matrix J, which are subject of the following set of conditions:

$$[B_1, B_2] + IJ = 0,$$

The solutions related by $GL(N)$ transformations

$$B_i' = gB_ig^{-1}, \;\; I' = gI, \;\; J' = Jg^{-1}; \quad g \in GL(N)$$

are equivalent.

The vectors obtained by the repeated action of B_1 and B_2 on $I_{1,2}$,columns of the matrix I ,span N-dimensional vector space V,a fiber of the N-dimensional fiber bundle ,whose base is the moduli space $\mathcal{M}_N$ itself.

The vector field and its fixed points.

The construction of the instanton partition function involves the determinants of the vector field v on $\mathcal{M}_N$, defined by

$$B_l \to t_l B_l; \quad I \to I t_v; \quad J \to t_1 t_2 t_v^{-1} J,$$

where parameters $t_l \equiv \exp \epsilon_l \tau$, $l = 1, 2$ and $t_v = \exp a\sigma_3 \tau$.

Fixed points, which are relevant for the determinants evaluation, are found from the conditions:

$$t_l B_l = g^{-1} B_l g; \quad I t_v = g^{-1} I; \quad t_1 t_2 t_v^{-1} J = J g.$$

The solutions of this system can be parameterized by pairs of Young diagrams $\vec{Y} = (Y_1, Y_2)$ such that the total number of boxes $|Y_1| + |Y_2| = N$. The cells $(i_1, j_1) \in Y_1$ and $(i_2, j_2) \in Y_2$ correspond to vectors $B_1^{i_1} B_2^{j_1} I_1$ and $B_1^{i_2} B_2^{j_2} I_2$ respectively. It is convenient to use these vectors as a basis in the fiber V attached to some fixed point.

Then the explicit form of the ADHM data for the given fixed point is defined straightaway

$$
\begin{aligned}
&g_{ss'} = \delta_{ss'} t_1^{i_s-1} t_2^{j_s-1}, \\
&(B_1)_{ss'} = \delta_{i_s+1,i_{s'}} \delta_{j_s,j_{s'}}, \\
&(B_2)_{ss'} = \delta_{i_s,i_{s'}} \delta_{j_s+1,j_{s'}}, \\
&(I_1)_s = \delta_{s,1}, \\
&(I_2)_s = \delta_{s,|Y_1|+1}, \\
&J = 0,
\end{aligned}
$$

where $s = (i_s, j_s)$.

Determinants of the vector field

The form of $\mathcal{N} = 2$ $SU(2)$ instanton partition function was derived to be equal an integral of the equivariantly form, defined in terms of the vector field v acting on the moduli space $\mathcal{M}_N$.

By localization technique , the moduli integral is reduced to the determinants of the vector field v in the vicinity of its fixed points

$$\mathcal{Z}_N(a, \epsilon_1, \epsilon_2) = \sum_n \frac{1}{\det_n v}.$$

We need to find all eigenvectors of the vector field on the tangent space passing through the fixed points

$$t_i \delta B_i = \Lambda\, g \delta B_i g^{-1},$$
$$\delta I t = \Lambda\, g \delta I,$$
$$t_1 t_2 t^{-1} \delta J = \Lambda\, \delta J g^{-1}.$$

This is equivalent to the following set of equations

$$\lambda\,(\delta B_i)_{ss'} = (\epsilon_i + \phi_{s'} - \phi_s)\,(\delta B_i)_{ss'},$$
$$\lambda\,(\delta I)_{sp} = (a_p - \phi_s)\,(\delta I)_{sp},$$
$$\lambda\,(\delta J)_{ps} = (\epsilon_1 + \epsilon_2 - a_p + \phi_s)\,(\delta J)_{ps},$$

where $\Lambda = \exp\lambda\tau$, $g_{ss} = \exp\phi_s\tau$ and

$$\phi_s = (i_s - 1)\epsilon_1 + (j_s - 1)\epsilon_2 + a_{p(s)}.$$

We should keep only those eigenvectors which belong to the tangent space. This means excluding variations breaking ADHM constraints. On the Moduli space

$$[\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J = 0.$$

Gauge symmetry can be taken into account in the following way. We fix a gauge in which $\delta B_{1,2}, \delta I, \delta J$ are orthogonal to any gauge transformation of $B_{1,2}, I, J$. This gives additional constraint

$$\left[\delta B_l, B_l^\dagger\right] + \delta I I^\dagger - J^\dagger \delta J = 0.$$

The variations in the LHS of the eq-ns above should be excluded.

The corresponding eigenvalues are defined from the equations

$$
\begin{aligned}
&t_1 t_2([\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J) = \\
&= \Lambda\, g \Big([\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J \Big) g^{-1}, \\
&\Big[\delta B_l, B_l^\dagger\Big] + \delta I I^\dagger - J^\dagger \delta J = \Lambda\, g \Big(\Big[\delta B_l, B_l^\dagger\Big] + \delta I I^\dagger - J^\dagger \delta J \Big) g^{-1}.
\end{aligned}
$$

One finds the following eigenvalues, which should be excluded :

$$
\begin{aligned}
\lambda &= (\epsilon_1 + \epsilon_2 + \phi_s - \phi_{s'}), \\
\lambda &= (\phi_s - \phi_{s'}).
\end{aligned}
$$

Thus, the determinant of the vector field is given by

$$
\det v = \frac{\prod\limits_{s,s' \in \vec{Y}} (\epsilon_1 + \phi_{s'} - \phi_s)(\epsilon_2 + \phi_{s'} - \phi_s) \prod\limits_{l=1,2; s \in \vec{Y}} (a_l - \phi_s)(\epsilon_1 + \epsilon_2 - a_l + \phi_s)}{\prod\limits_{s,s' \in \vec{Y}} (\phi_{s'} - \phi_s)(\epsilon_1 + \epsilon_2 - \phi_{s'} + \phi_s)}
$$

Modified moduli space

The subspace of the Moduli space $\mathcal{M}_{\text{sym}}$ for $SU(2)$ gauge group is defined by the following additional restriction of $\mathbb{Z}_2$ symmetry

$$B_1 = -PB_1P^{-1}; B_2 = -PB_2P^{-1}; \qquad I = PI; \qquad J = JP^{-1}.$$

where $P \in GL(N)$ is some gauge transformation, obviously $P^2 = 1$.

New manifold $\mathcal{M}_{\text{sym}}$ is a disjoint union of connected components $\mathcal{M}_{\text{sym}}(N_+, N_-)$, where N_+ and N_- are integers which denote the dimensions of V_+ and V_- (*i.e.* even and odd subspaces of the fiber V) , $N_+ + N_- = N$. These numbers are fixed inside a given connected component of $\mathcal{M}_{\text{sym}}$. Each component is connected and can be considered separately.

Returning to to $\mathcal{M}_{\text{sym}}$ we note that it contains all fixed points of the vector field found above. For the operator P in the fixed point $\vec{Y}$ we get

$$P(B_1^{i-1}B_2^{j-1}I_\alpha) = (-1)^{i+j}B_1^{i-1}B_2^{j-1}I_\alpha,$$

so its matrix elements can be found explicitly, $P_{ss'} = (-1)^{i_s+j_s}\delta_{ss'}$ The parity characteristic $P(s) = (-1)^{i_s+j_s}$ is assigned to each box in the Young diagrams related to the fixed point. We use convenient notation that each box is white or black. If $P(s) = 1$, the box is white, and if $P(s) = -1$,it is black.

Therefore the fixed points can be classified by the numbers of white and black boxes, N_+ and N_-. These numbers are equal to the dimensions of the subspaces V_+ and V_- of the fiber attached to those points of $\mathcal{M}_{\text{sym}}$ which belong to the same component as the fixed point itself.

Modified moduli space and $\widehat{gl}(2)_2 \times \mathcal{NSR}$ algebra

The Whittaker vector found used not the whole space $\mathcal{M}_{\textsf{sym}}$, but only its connected components $\mathcal{M}_{\textsf{sym}}(N,N)$ and $\mathcal{M}_{\textsf{sym}}(N,N-1)$. The norm of the Whittaker vector is equal to the sum of contributions of fixed points. In this section we calculate the number of fixed points on such components and discuss the result from the $\widehat{gl}(2)_2 \times \mathcal{NSR}$ point of view.

We introduce the generating function

$$\chi(q) = \sum_N |\mathcal{M}_{\textsf{sym}}(N,N)| q^N + \sum_N |\mathcal{M}_{\textsf{sym}}(N,N-1)| q^{N-1/2},$$

where $|\mathcal{M}_{\textsf{sym}}(N_+,N_-)|$ is a number of fixed points on $\mathcal{M}_{\textsf{sym}}(N_+,N_-)$. This number equal to the number of pairs of Young diagrams with N_+ white boxes and N_- black boxes.

Denote by $d(Y) = N_+(Y) - N_-(Y)$ the difference between number of white and black boxes in Young diagram Y. For any integer k we denote by

$$\chi_k(q) = \sum_{d(Y)=k} q^{\frac{|Y|}{2}},$$

the generating function of Young diagrams of given difference $d(Y)$. This function has the form:

$$\chi_k(q) = q^{\frac{2k^2-k}{2}} \prod_{m\geq 0} \frac{1}{(1-q^{m+1})^2}.$$

for $k = 0$. The factor $q^{\frac{2k^2+k}{2}}$ corresponds to the smallest Young diagram with $d(Y) = k$. For $k > 0$ this diagram consist of $2k-1$ rows of length $2k-1, 2k-2, \ldots, 1$. For $k < 0$ this diagram consist of $2|k|$ rows of length $2|k|, 2|k|-1, \ldots, 1$.

The generating function of pairs Young diagrams with $N_+ - N_- = k$ reads

$$\chi_k^{(2)} = \sum_{k_1+k_2=k} \chi_{k_1}\chi_{k_2},$$

Using () and Jacobi triple product identity

$$\sum_{n\in\mathbb{Z}} (-1)^n t^n q^{n^2} = \prod_{m\geq 0} (1-q^{2m+2})(1-q^{2m+1}t)(1-q^{2m+1}t^{-1})$$

we get

$$\chi(q) = \chi_0^{(2)}(q) + \chi_1^{(2)}(q) = \prod_{m\geq 0} \frac{(1-q^{2m+1})^2}{(1-q^{2m+2})^3} = \chi_B(q)^3 \chi_F(q)^2,$$

where

$$\chi_B(q) = \prod_{n\in\mathbb{Z},\, n>0} \frac{1}{(1-q^n)}$$

$$\chi_F(q) = \prod_{r\in\mathbb{Z}+\frac{1}{2},\, r>0} (1+q^r).$$

The first terms of the series for $\chi(q)$ looks as follows

$$\chi(q) = 1 + 2q^{1/2} + 4q + 8q^{3/2} + 16q^2 + 28q^{5/2} + ...$$

The $\chi_B(q)\chi_F(q)$ equals to the character of standard representation of the $\mathcal{NSR}$ algebra with generators L_n, G_r. $\chi_B(q)$ equals to the character of the Fock representation of the Heisenberg algebra. The term $\chi_B(q)\chi_F(q)$ should be related to the fact that $\widehat{sl}(2)$ representation of level 2 can be realized by one bosonic and one fermionic field .

The the generating function the whole space $\mathcal{M}_{\text{sym}}$ has the form

$$\chi(q) = \sum_N |\mathcal{M}_{\text{sym}}(N)| q^{\frac{N}{2}} = \prod_{n \in \mathbb{Z}, n > 0} \frac{1}{\left(1 - q^{\frac{n}{2}}\right)^2}$$

The result equals to the character of the simple representation of $\widehat{gl}(2)_2 \times \mathcal{NSR}$ namely the tensor product of Fock representation of Heisenberg algebra, vacuum representation of $\widehat{gl}(2)_2$ and NS representation of $\mathcal{NSR}$.

Determinants of the vector field for $\mathcal{M}_{\text{sym}}$

The tangent space for this case is reduced by the additional requirement

$$-\delta B_{1,2} = P\delta B_{1,2}P^{-1}; \qquad \delta I = P\delta I; \qquad \delta J = \delta J P^{-1},$$

or, on the level of the matrix elements,

$$-(\delta B_{1,2})_{ss'} = P(s)(\delta B_{1,2})_{ss'}P(s'); \quad (\delta I)_{sp} = P(s)(\delta I)_{sp};$$

$$(\delta J)_{ps} = (\delta J)_{ps}P(s),$$

The first relation means that only eigenvectors $(\delta B_{1,2})_{ss'}$ with the different colors of s and s' belong to $\mathcal{M}_{\text{sym}}$. Similarly, the second and third leave $(\delta J)_{ps}$and $(\delta J)_{ps}$ only if s is white. Thus, we get the new determinant $\det{}'v$

$$\frac{\prod\limits_{\substack{s,s'\in\vec{Y}\\ P(s)\neq P(s')}} (\epsilon_1+\phi_{s'}-\phi_s)(\epsilon_2+\phi_{s'}-\phi_s) \prod\limits_{\substack{\alpha=1,2;s\in\vec{Y}\\ P(s)=1}} (a_\alpha-\phi_s)(\epsilon_1+\epsilon_2-a_\alpha+\phi_s)}{\prod\limits_{\substack{s,s'\in\vec{Y}\\ P(s)=P(s')}} (\phi_{s'}-\phi_s)(\epsilon_1+\epsilon_2-\phi_{s'}+\phi_s)}$$

Re-expressed in terms of arm-length and leg-length this expression gives

$$\det{}' v = \prod_{\alpha,\beta=1}^{2} \prod_{s \in {}^{\diamondsuit}Y_\alpha(\beta)} E\Big(a_\alpha - a_\beta, Y_\alpha, Y_\beta \Big| s\Big)(Q - E\Big(a_\alpha - a_\beta, Y_\alpha, Y_\beta \Big| s\Big)),$$

here $E\Big(a, Y_1, Y_2 \Big| s\Big)$ are defined as follows

$$E\Big(a, Y_1, Y_2 \Big| s\Big) = a + b(L_{Y_1}(s) + 1) - b^{-1} A_{Y_2}(s) \ ,$$

where $A_Y(s)$ and $L_Y(s)$ are respectively the arm-length and the leg-length for a cell s in Y. The region ${}^{\diamondsuit}Y_\alpha(\beta)$ is defined as

$${}^{\diamondsuit}Y_\alpha(\beta) = \Big\{(i,j) \in Y_\alpha \Big| P\Big(k'_j(Y_\alpha)\Big) \neq P\Big(k_i(Y_\beta)\Big)\Big\},$$

or, in other words, the boxes having different parity of the leg- and arm-factors. So the contribution of the vector multiplet reads

$$Z^{\text{sym}}_{\text{vec}}(\vec{a}, \vec{Y}) \equiv \frac{1}{\det{}' v}$$

Matter multiplets

The hypermultiplets with masses μ give some additional contribution because of appearence of the N fermionic null-modes . The amplitudes ψ of the null-modes can be considered as of the fiber V attached to one of the fixed point $\vec{Y}$. The eigenvalues of the vector field are defined from the equation

$$\lambda\,\psi_s = (\mu + \phi_s)\,\psi_s,$$

The corresponding contribution of the fundamental hypermultiplets with masses μ_i looks as follows

$$Z_{\mathsf{f}}(\mu_i, \vec{a}, \vec{Y}) = \prod_{i=1}^{4} \prod_{\alpha=1}^{2} \prod_{s \in Y_\alpha} \Big(\phi(a_\alpha, s) + \mu_i\Big),$$

Considering the case of $\mathcal{M}_{\mathsf{sym}}$ we impose some restrictions on the set of eigenvectors for the fundamental multiplets. $\psi \in V_+$,if N-even and $\psi \in V_+$, if N-odd.

The above consideration suggests the following form of the contributions of the fundamental hyper multiplets

$$Z_{\mathrm{f}}^{\mathrm{sym}(0)}(\mu_i,\vec{a},\vec{Y}) = \prod_{i=1}^{4}\prod_{\alpha=1}^{2}\prod_{s\in Y_\alpha, s-\mathrm{white}}\Big(\phi(a_\alpha,s)+\mu_i\Big),$$

$$Z_{\mathrm{f}}^{\mathrm{sym}(1)}(\mu_i,\vec{a},\vec{Y}) = \prod_{i=1}^{4}\prod_{\alpha=1}^{2}\prod_{s\in Y_\alpha, s-\mathrm{black}}\Big(\phi(a_\alpha,s)+\mu_i\Big),$$

The first expression correspond to the case with even number of instantons ,the second one correspond to the case with odd number of instantons.

Four-point Super Liouville conformal block

Two-dimensional super conformal Liouville field theory arises in non-critical String theory. The Lagrangian of the theory reads

$$\mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi}(\partial_a\phi)^2 + \frac{1}{2\pi}\left(\psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi}\right) + 2i\mu b^2\bar{\psi}\psi e^{b\phi} + 2\pi b^2\mu^2 e^{2b\phi} .$$

Here μ is the cosmological constant and parameter b is related to the central charge c of the super-Virasoro algebra

$$c = 1 + 2Q^2 , \ \ Q = b + \frac{1}{b} .$$

We are interested in the Neveu-Schwarz sector of the super-Virasoro algebra

$$[L_m, L_n] = (n-m)L_{n+m} + \frac{c}{8}(n^3 - n)\delta_{n+m} ,$$
$$\{G_r, G_s\} = 2L_{r+s} + \frac{1}{2}c(r^2 - \frac{1}{4})\delta_{r+s} ,$$
$$[L_n, G_r] = (\frac{1}{2}n - r)G_{n+r} .$$

where the subscripts m, n –integers and r, s – half-integers. The NS fields belong to highest weight representations of super-Virasoro algebra.

The central problems in CFT is the computation of the correlation functions of the primary fields Φ_Δ and Ψ_Δ has the conformal dimension Δ defined by $L_0|\Delta\rangle = \Delta|\Delta\rangle$, while $\Psi_\Delta \equiv G_{-1/2}\Phi_\Delta$. Together fields Φ_Δ and Ψ_Δ form primary super doublet. The standart parametrization of the conformal dimensions

$$\Delta(\lambda) = \frac{Q^2}{8} - \frac{\lambda^2}{2}.$$

4-point correlation function of bosonic primaries Φ_i is expressed in terms superconformal blocks

$$\langle \Phi_1(q)\Phi_2(0)\Phi_3(1)\Phi_4(\infty)\rangle =$$

$$(q\bar{q})^{\Delta-\Delta_1-\Delta_2} \sum_\Delta \Big(C_{12}^{\Delta} C_{34}^{\Delta} F_0(\Delta_i|\Delta|q) F_0(\Delta_i|\Delta|\bar{q}) + \tilde{C}_{12}^{\Delta} \tilde{C}_{34}^{\Delta} F_1(\Delta_i|\Delta|q) F_1(\Delta_i|\Delta|\bar{q}) \Big) .$$

The first few coefficients of the superconformal blocks $F_{0,1}$

$$F_0(\Delta_i|\Delta|q) = \sum_{N=0,1,\dots} q^N F^{(N)}(\Delta_i|\Delta) \ ,$$

$$F_1(\Delta_i|\Delta|q) = \sum_{N=1/2,3/2,\dots} q^N F^{(N)}(\Delta_i|\Delta) \ ,$$

$$F^{(0)} = 1,$$

$$F^{(\frac{1}{2})} = \frac{1}{2\Delta},$$

$$F^{(1)} = \frac{(\Delta + \Delta_1 - \Delta_2)(\Delta + \Delta_3 - \Delta_4)}{2\Delta}$$

$$F^{(\frac{3}{2})} = \frac{(1 + 2\Delta + 2\Delta_1 - 2\Delta_2)(1 + 2\Delta + 2\Delta_3 - 2\Delta_4)}{8\Delta(1 + 2\Delta)}$$

$$+ \frac{4(\Delta_1 - \Delta_2)(\Delta_3 - \Delta_4)}{(1 + 2\Delta)(c + 2(-3 + c)\Delta + 4\Delta^2)},$$

We suggest the new representation for the NS four-point conformal blocks :

$$\sum_{N=0,1,\ldots} q^N \sum_{\vec{Y},\, \substack{N_+(\vec{Y})=N \\ N_-(\vec{Y})=N}} \frac{Z_{\mathsf{f}}^{\mathsf{sym}(0)}(\mu_i, \vec{a}, \vec{Y})}{Z_{\mathsf{vec}}^{\mathsf{sym}}(\vec{a}, \vec{Y})} = (1-q)^A F_0(\Delta(\lambda_i)|\Delta(a)|q)$$

$$\sum_{N=\frac{1}{2},\frac{3}{2},\ldots} q^N \sum_{\vec{Y},\, \substack{N_+(\vec{Y})=N+\frac{1}{2} \\ N_-(\vec{Y})=N-\frac{1}{2}}} \frac{Z_{\mathsf{f}}^{\mathsf{sym}(1)}(\mu_i, \vec{a}, \vec{Y})}{Z_{\mathsf{vec}}^{\mathsf{sym}}(\vec{a}, \vec{Y})} = (1-q)^A F_1(\Delta(\lambda_i)|\Delta(a)|q) \,.$$

The formula is the main result of this talk. The parameters of the conformal block are related to those of the instanton partition function as

$$\mu_1 = \frac{Q}{2} - (\lambda_1 + \lambda_2), \qquad \mu_2 = \frac{Q}{2} - (\lambda_1 - \lambda_2),$$
$$\mu_3 = \frac{Q}{2} - (\lambda_3 + \lambda_4), \qquad \mu_4 = \frac{Q}{2} - (\lambda_3 - \lambda_4),$$

and

$$A = \left(\frac{Q}{2} - \lambda_1\right)\left(\frac{Q}{2} - \lambda_3\right).$$