Gravitational Four-Fermion Interaction and Hamiltonian of Ferromagnetic Type

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Action (almost common one) for gravitational field is (Holst)

$$S_g = -\frac{1}{16\pi G}\int d^4x\, e\, e^{\mu}_{I}\, e^{\nu}_{J}\left(R^{IJ}_{\mu\nu} - \frac{1}{\gamma}{}^*R^{IJ}_{\mu\nu}\right).$$

Here $R^{IJ}_{\mu\nu} = -\partial_\mu \omega^{IJ}{}_{\nu} + \partial_\nu \omega^{IJ}{}_{\mu} + \omega^{IK}{}_{\mu}\omega_{K}{}^{J}{}_{\nu} - \omega^{IK}{}_{\nu}\omega_{K}{}^{J}{}_{\mu}$,

${}^*R^{IJ}_{\mu\nu} = \frac{1}{2}\varepsilon^{IJ}_{KL}R^{KL}_{\mu\nu}$,

Barbero - Immirzi parameter $\gamma = 0.274$ is solution of "secular" equation

$$\sum_{j=1/2}^{\infty}(2j+1)e^{-2\pi\gamma\sqrt{j(j+1)}} = 1.$$

Interaction of fermions with gravity results in four-fermion interaction of axial currents A^I (Kibble; Rovelli)

$$S_A = -\frac{3}{2}\pi G\,\frac{\gamma^2}{\gamma^2+1}\int d^4x\, e\ \eta_{IJ}A^I A^J.$$

Common action for fermions in gravitational field can be generalized as follows:

$$S_f = \int d^4x\, e\,\frac{1}{2}[(1-i\alpha)\,\bar{\psi}\,\gamma^I\,e^\mu_I\,i\nabla_\mu\psi - (1+i\alpha)\,i\,\overline{\nabla_\mu\psi}\,\gamma^I e^\mu_I\psi];$$

$$\nabla_\mu = \partial_\mu - \frac{1}{4}\omega^{IJ}{}_\mu\gamma_I\gamma_J, \quad [\nabla_\mu, \nabla_\nu] = \frac{1}{4}R^{IJ}_{\mu\nu}\gamma_I\gamma_J.$$

Real constant α becomes operative in the presence of so-called contorsion tensor. It modifies gravitational four-fermion interaction to (Freidel et al)

$$S_{ff} = -\frac{3}{2}\pi\, G\frac{\gamma^2}{\gamma^2+1}\int d^4x\, e\,\Big[\eta_{IJ}A^I A^J - \frac{\alpha}{\gamma}\,\eta_{IJ}(V^I A^J + A^I V^J) - \alpha^2\,\eta_{IJ}V^I V^J\Big]$$

This interaction, proportional to Newton constant G and to particle number density ρ squared, gets essential and dominating on the Planck scale only. Here, due to pair creation, number densities of fermions and antifermions increase, but vector current density V^I remains the same. Axial current density A^I is C-even, fermions and antifermions contribute to A^I with same sign, so that A^I grows up together with density and temperature.

Thus, in Planckian and sub-Planckian regimes (the only ones, where gravitational 4-fermion interaction can be essential), V- dependent contributions can be neglected, and 4-fermion interaction reduces to axial-axial one:

$$S_A = -\frac{3}{2}\pi G \frac{\gamma^2}{\gamma^2+1}\int d^4x\, e\;\eta_{IJ}A^I A^J.$$

Now, analysis of this expression.

In center-of-mass system, axial current of fermion a is

$$A^I_a = \frac{1}{4}\phi^\dagger_a\{\sigma_a(\mathbf{n}' + \mathbf{n}),\ (1 - (\mathbf{n}'\mathbf{n}))\sigma_a + \mathbf{n}'(\sigma_a\mathbf{n}) + \mathbf{n}(\sigma_a\mathbf{n}') - i[\mathbf{n}' \times \mathbf{n}]\}\phi_a;$$

$\mathbf{n}$ and $\mathbf{n}'$ are unit vectors of initial and final momenta, respectively.
Averaging over directions of $\mathbf{n}$ and $\mathbf{n}'$, results in

$$< \eta_{IJ} A^I_a A^J_b >= \frac{1}{72}(3 - 11\sigma_a\sigma_b)\,.$$

Corresponding effective two-particle spin-spin Hamiltonian is

$$\boldsymbol{H_{ab} = \frac{1}{48}\,\pi\,G\,\frac{\gamma^2}{\gamma^2+1}\,[\,3 - 11(\sigma_a\sigma_b)]\,\delta(\mathbf{r}_a - \mathbf{r}_b)}\,.$$

This interaction takes place in $\boldsymbol{S}$-state only. For identical fermions (antifermions) scattering occurs in singlet spin state, where $\boldsymbol{(\sigma_{1a}\sigma_{2a}) = -3}$,
thus their Hamiltonian has no spin degrees of freedom at all:

$$\boldsymbol{H_{aa} = \frac{3}{4}\,\pi\;G\,\frac{\gamma^2}{\gamma^2+1}\;\delta(\mathbf{r}_{1a} - \mathbf{r}_{2a})}\,.$$

For nonidentical fermions, we obtain effective spin Hamiltonian of ferromagnetic type:

$$H_{ab} = \frac{1}{96}\,\pi\, G\,\frac{\gamma^2}{\gamma^2+1}\sum_{a\neq b}\left[\,3 - 11(\sigma_a\sigma_b)\,\right]\delta(\mathbf{r}_a - \mathbf{r}_b)\,.$$

Since number of fermionic species is large, this interaction is dominating.

Hence following estimate for "ferromagnetic" energy ε per fermion:

$$\varepsilon \sim\ -G\,\rho\,;$$

here ρ is total number density of fermions and antifermions.

On Planck scale, with $\rho \sim m_{\rm Pl}^3$ ($m_{\rm Pl}$ is Planck mass), this "ferromagnetic" energy per fermion

$$\varepsilon \sim -m_{\rm Pl}$$

is on the same order of magnitude as temperature τ, plausible estimate for the latter being

$$\tau \sim \rho^{1/3} \sim m_{\rm Pl}\,.$$

With $|\varepsilon| \sim \rho$ and $\tau \sim \rho^{1/3}$, at sufficiently high densities discussed "ferromagnetic" ordering cannot be destroyed by thermal effects.

The state with negative ''ferromagnetic'' energy density, proportional to ρ^2, is unstable wrt further compression. No physical effect, that could stop this collapse, is known. Thus, it is natural to assume that

sub-Planckian stage had not existed at all

At last, on energy-momentum tensor (EMT) of the discussed system. For system of point-like particles with electromagnetic interaction among them, trace T_μ^μ of its EMT satisfies condition

$$T \equiv T_\mu^\mu = g_{\mu\nu} T^{\mu\nu} \geq 0\,.$$

Common assumption is that this condition is valid for all interactions existent in the Nature. However, in our problem, Lagrangian density

$$L = -\frac{3}{2}\pi G \,\frac{\gamma^2}{\gamma^2+1}\, e\;\eta_{IJ} A^I A^J$$

is independent of derivatives. Therefore its EMT reduces to

$$T^{IJ} = -\eta^{IJ} L.$$

Its trace equals

$$T = -4\,L = 4\,T^{00} = 4\,H,$$

here H is Hamiltonian of system. In ferromagnetic phase, this Hamiltonian is negative. Therefore, here trace of EMT is negative as well:

$$T < 0.$$

Obviously, this unusual property of discussed EMT is directly related to above mentioned instability of discussed system with respect to gravitational compression.